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THE DYNAMICS OF SYSTEMS NEAR TO GRAZING COLLISION†

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Families of motions of a system of rigid bodies that include collisions with arbitrarily small (including zero) initial velocity of approach are considered. The formal description of such collisions using the stereomechanical axiom leads to dynamical paradoxes. Another solution to the problem of grazing collision is proposed, based on the visco-elastic model of contact stresses.

1. INTRODUCTION

We specify generalized coordinates $\mathbf{q}=(q_0, q_1, \ldots, q_n)$ such that q_0 is the distance between the colliding pair of elements. Then the quantities $q_1, \ldots, q_n$ are arbitrary whereas the first coordinate is constrained by the single-sided constraint $q_0 \geqslant 0$. In the domain $q_0>0$ the motion is described by the equations

$$\ddot{\mathbf{q}}=\mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{1.1}$$

To describe collisions it is usual to use the formally-axiomatic stereomechanical collision theory [1], where one ignores the collision duration τ and also the role of "active" forces $\mathbf{Q}$ compared with the contact stresses in the formation of a collisional impulse. The stereomechanical equations are

$$\mathbf{q}^+=\mathbf{q}^-, \quad \dot{\mathbf{q}}^+=\mathbf{U}(\mathbf{q}^-, \dot{\mathbf{q}}^-) \tag{1.2}$$

where the minus and plus signs correspond to the start and end of the collision, and $\mathbf{U}$ is some map from the half-space $\dot{q}_0<0$ into the half-space $\dot{q}>0$. In particular, for a frictionless collision, with a suitable choice of generalized coordinates [2] the map $\mathbf{U}$ has the form

$$\dot{\mathbf{q}}_0^+=-\kappa\dot{\mathbf{q}}_0^-, \quad \dot{\mathbf{q}}_j^+=\dot{\mathbf{q}}_j^- \; (j=1,2,\ldots,n) \tag{1.3}$$

where $\kappa\in[0, 1]$ is the Newtonian coefficient of restitution with respect to the velocity.

A contradiction appears in Eqs (1.2) for motions in which the constraint is touched. Suppose that at some time $t=t^*$ the conditions

$$q_0=0, \quad \dot{q}_0=0, \quad F_0>0 \tag{1.4}$$

are satisfied.

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The inequality in (1.4) ensures the weakening of the single-sided constraint: $q_0 > 0$ when $t = t^*$. In this case arbitrarily small changes to the initial conditions can have two kinds of consequence: either the bodies do not come in contact at all near the time $t = t^*$, or they collide with an arbitrarily small initial velocity. We will discuss the two situations.

1. For rough bodies not possessing spherical symmetry, a "grazing" collision is possible: the magnitude of the collision impulse does not vanish when $\dot{q}_0(t^*) \to -0$ [3, 4]. Here the motion does not depend continuously on the initial conditions [5].

2. For bodies with totally smooth surfaces the collision is described by Eqs (1.3). Suppose that the system has a T-periodic motion $\mathbf{q}^*(t)$ that includes a grazing collision (1.4). To this motion there corresponds a fixed point of the mapping of the phase space into itself along integral curves between times $t = t^*$ and $t = t^* + T$. In a neighbourhood of the fixed point this mapping has an unbounded derivative as $\tau \to 0$ [6]. Such a situation is outside the framework of linear theory and is known as C-bifurcation [7]. In C-bifurcation the multipliers are discontinuous functions of the parameters and a general approach to its investigation has not yet been developed. (Various special cases [7–9] have been studied.)

A satisfactory solution to these problems based on the stereomechanical equations (1.2) does not appear to be possible. Below we present a more appropriate grazing collision model and use it to solve these problems.

2. INVESTIGATION OF THE GRAZING COLLISION OF A PARTICLE WITH A VISCO-ELASTIC MEDIUM

We suppose that the contact stresses are determined by the properties of a Kelvin–Voight medium: the inequality $q_0 < 0$ is possible, and it is accompanied by the appearance of a reaction R_0 linear with respect to q_0 and $\dot{q}_0$ [1]. The equations of motion in the domain $q_0 < 0$ have the form

$$\ddot{q}_0 = -M^2 q_0 - 2\alpha M \dot{q}_0 + F_0, \quad \ddot{q}_j = F_j \quad (j = 1, \dots, n), \alpha \in (0,1) \tag{2.1}$$

where the value of the coefficient M, describing the contact stiffness, is large. The collision corresponds to the part of the trajectory of system (2.1) with initial conditions

$$q_0(t^*) = 0, \quad q_i(t^*) = q_i^0, \dot{q}_0(t^*) = -u < 0, \ \dot{q}_i(t^*) = \dot{q}_i^0 \tag{2.2}$$

over the interval $t \in (t^*, \ t^* + \tau)$, where τ is the smallest positive root of the equation $q_0(t^* + \tau) = 0$.

An analytic solution to system (2.1) is not in general possible, but approximate solutions enable us to draw conclusions about the collision mechanism. A case where the quantity u was fixed was investigated, and it was shown [10] that the estimates

$$\tau = \pi\delta^{-1}M^{-1} + o(M^{-1}), \dot{q}_0(t^* + \tau) = \kappa u + O(M^{-1}) \tag{2.3}$$

$$\dot{q}_i(t^* + \tau) = \dot{q}_i^0 + O(M^{-1}) \quad (i = 1, \dots, n)$$

$$\delta = (1 - \alpha^2)^{1/2}, \quad \kappa = \exp(-\pi\alpha / \delta)$$

hold irrespective of the form of the function $\mathbf{F}$.

Relations (2.3) in the limit as $M \to \infty$ become the stereomechanical equations (1.3), and this verifies the use of model (2.1) in describing collisions.

In the neighbourhood of a grazing condition the values of u can be as small as desired, which corresponds to another limiting case of the solution: $u \to 0$ and fixed M. We note that according to (1.4)

$$C_0 = F_0(t^*, 0, q_i^0, 0, \dot{q}_i^0)$$

is positive; we choose u to be so small that $Mu/C_0 = \chi \ll 1$. Using the initial conditions (2.2) the solution to the first of Eqs (2.1) can be represented as a Taylor series

$$q_0(t^* + \Delta t) = -u\Delta t + \tfrac{1}{2}(C_0 + O(\chi))(\Delta t)^2 + \vartheta(\Delta t)^2 \tag{2.4}$$

Equating the right-hand side of (2.4) to zero, we find the duration τ of the collision, and then the final values of the variables

$$\tau = 2u / C_0 + o(u^2), \quad \dot{q}_0(t^* + \tau) = u + o(u) \tag{2.5}$$

$$p_i(t^* + \tau) = p_i^0 + O(u) \quad (i = 1, \ldots, n)$$

Comparing the two groups of equations (2.3) and (2.5), we note properties of collisions with small initial velocities of approach: firstly, the duration of such collisions decreases with u, secondly, the coefficient of restitution κ is close to unity, and thirdly, as $u \to 0$ the dominant component in the first formula of (2.1) is the C_0 term, that is insignificant in relations (2.3).

This last property is obtained by analysing formula (2.4)

$$q_0(t^* + \Delta t) \geqslant -\tfrac{1}{2}u^2 C_0^{-1} + o(u^2), \quad \dot{q}_0(t^* + \Delta t) \geqslant -u$$

$$-M^2 q_0 - 2\alpha M\dot{q}_0 \leqslant C_0(2\alpha\chi + \tfrac{1}{2}\chi^2 + o(\chi^2)), \quad F_0 = C_0 + O(\chi)$$

Remark. 1. If the dependence of the reaction R_0 on $\mathbf{q}$ and $\dot{\mathbf{q}}$ differs from (2.1), but $R_0 \to 0$ as q_0, $\dot{q}_0 \to 0$, then for sufficiently small u the collision is described by the same formulae (2.5) because in their derivation the explicit form of the function R_0 is not used.

2. Relations (2.5) also remain valid for collisions between bodies with rough surfaces. Here Eq. (2.1) becomes more complicated because of the presence of grazing reaction components [11]. Nevertheless, as in the previous remark, all estimates and the basic conclusion remain true: for sufficiently small u the motion is basically determined by the active forces and not by the reaction.

In this model the first of the paradoxes noted in the Introduction is resolved: in all cases the collisional impulse vanishes as $u \to +0$.

3. ANALYSIS OF THE C-BIFURCATION

We will now solve the second of the problems posed in the Introduction. We first suppose that $n = 0$, i.e. the given mechanical system has one degree of freedom. We represent the equations of motion in the form (the index is omitted)

$$\ddot{q} = F + R, \; F = F(\mu, t, q, \dot{q}), \; R = \begin{cases} 0 & q > 0 \\ -M^2 q - 2\alpha M\dot{q}, & q < 0 \end{cases} \tag{3.1}$$

where μ is a parameter. When $\mu = 0$ system (3.1) has a T-periodic solution $q_0^*(t)$, including a grazing collision (1.4) and, possibly, other collisions described by (1.3). Solutions of system (3.1) define the point mapping $\Phi_\mu : (q(t^*),\ \dot{q}(t^*)) \to (q(t^* + T),\ \dot{q}(t^* + T))$, with $\Phi_0(0,\ 0) = (0,\ 0)$. Because the reaction R is continuous at the origin, the mapping Φ_0 is differentiable

$$\Phi_0(q, \dot{q}) = X_0 \left\| \begin{matrix} q \\ \dot{q} \end{matrix} \right\| + O(q^2 + \dot{q}^2)$$

In this formula the defining matrix X_0 is equal to the value at $t=t^*+T$ of the fundamental matrix of solutions $X(t^*, t)$ of the variational equations

$$\dot{X}(t^*,t)=\begin{Vmatrix} 0 & 1 \\ F_q' & F_{\dot{q}}' \end{Vmatrix} X(t^*,t), \quad X(t^*,t^*)=E_2 \tag{3.2}$$

and here, at the times t_k when the periodic motion $q_0^*(t)$ undergoes collisions, the matrix $X(t^*, t)$ changes abruptly according to the formula [12]

$$X(t^*,t_k+0)=\begin{Vmatrix} -\kappa & 0 \\ b_{21} & -\kappa \end{Vmatrix} X(t^*,t_k) \tag{3.3}$$

$$b_{21}=[F(\mu,t_k,0,-\kappa\dot{q}_0^*(t_k))+\kappa F(\mu,t_k,0,\dot{q}_0^*(t_k))]/\dot{q}_0^*(t_k)$$

If unity is not one of the eigenvalues of the matrix X_0, then according to the general theory of point mappings [13], for sufficiently small μ the mapping Φ_μ has a fixed point which becomes the origin as $\mu\to 0$. In the first approximation the coordinates $(\xi(\mu), \eta(\mu))$ of this point are found from the equation

$$(E_2-X_0)\begin{Vmatrix} \xi(\mu) \\ \eta(\mu) \end{Vmatrix} = \left.\frac{\partial\Phi_\mu(0,0)}{\partial\mu}\right|_{\mu=0}$$

In the general case $d\xi(\mu)/d\mu\neq 0$, and the quantity $\xi(\mu)$ changes sign in the neighbourhood of zero. This gives the essence of a C-bifurcation: for $\mu<0$ the motion $q_\mu^*(t)$ departs from collision in the neighbourhood of the time t^*, while for $\mu>0$ it experiences collisions with an initial velocity that vanishes as $\mu\to+0$ (or vice versa).

The evolution of the motion $q_\mu^*(t)$ in the regular domain $\mu<0$ is described by the usual theory, because in formula (3.3) the denominator does not vanish. In the region $\mu<0$ where grazing collisions exist one cannot use this formula because $b_{21}\to\infty$ as $\dot{q}^*\to 0$. One can instead directly vary system (3.1) in the domain $q<0$. Equation (3.2) then acquires the form

$$\dot{X}(t^*,t)=\left(\begin{Vmatrix} 0 & 1 \\ -M^2 & -2\alpha M \end{Vmatrix}+O(1)\right)X(t^*,t)$$

where the quantity $O(1)$ is bounded as $M\to\infty$. Using formula (2.5) for the monodromy matrix when $\mu>0$ we obtain

$$X_\mu=\left(\begin{Vmatrix} 1 & 0 \\ -M_\chi^2 & 1 \end{Vmatrix}+O(\chi)\right)X_0, \quad \chi=M\left[-\frac{2\xi(\mu)}{C_0}\right]^{1/2}\ll 1 \tag{3.4}$$

We denote by $\rho_{1,2}(\mu)$ the eigenvalues of the matrix X_μ (multipliers) and assume that $|\rho_{1,2}(0)|<1$ (i.e. the periodic motion is asymptotically stable for $\mu\leqslant 0$). For sufficiently small values of $\mu>0$, such that $\chi=O(M^{-1})$, the determinant of the matrix X_μ is equal to $\rho_1(0)\rho_2(0)$, and its trace is equal to $-\rho_1(0)-\rho_2(0)-M\chi x_{12}$, where $x_{ij}(i,\ j=1,\ 2)$ are the elements of the matrix X_0. As χ increases one of the multipliers remains inside the unit circle, and the second leaves it along the real axis, becoming plus or minus one depending on the sign of x_{12}. The following assertion therefore holds.

Theorem. If $x_{12}<0$, then the C-bifurcation is of the "saddle-node" type: the given periodic motion merges with some unstable motion of the same period and both vanish. In the $x_{12}>0$ case this bifurcation leads to a sequence (finite or infinite) of period doublings.

Remark. 1. If any period-doubling bifurcation is subcritical, then it is the last one in the sequence referred to above. Supercritical bifurcations form an infinite sequence, leading to chaos at $\mu = O(M^{-4})$. In both cases stable periodic or subperiodic motions are not preserved.

2. The $x_{12} = 0$ case is singular: a stable periodic motion $q_\mu^*(t)$ can be preserved under a C-bifurcation [12]. Moreover, for values of x_{12} close to zero, a C-bifurcation may interact with another "saddle-node" bifurcation [14]. As a result, a new stable periodic motion is generated, which turns into the original one as $x_{12} \to 0$.

In the general case $n > 0$, by analogy with formula (3.4), for small values of $\mu > 0$ the matrix X_0 is multiplied by the matrix

$$\begin{Vmatrix} 1 & 0 & 0 & \dots & 0 \\ -M\chi & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{Vmatrix}$$

As μ increases from zero the multipliers $\rho_j(\mu)(j = 1, 2, \dots, 2n+2)$ vary continuously and the C-bifurcation decomposes into simpler types. Alongside the "saddle-nodes" and "forks" considered above, here we also have the possibility of a Hopf bifurcation, as a result of which a family of quasi-periodic motions is created [15]. As μ increases further these families are in general disrupted and the system becomes chaotic.

4. EXAMPLE OF A C-BIFURCATION

Consider the forced oscillations of a linear oscillator with a stop (Fig. 1).

$$\ddot{q} + 2k\dot{q} + c^2 q = Q(\mu, t),\ q \geqslant 0, \quad k \geqslant 0 \tag{4.1}$$

$$Q(\mu, t) = c^2(1-\mu) + (1-c^2)\cos t + 2k \sin t$$

In this problem the use of Eq. (4.1) means that the barrier S is replaced by a spring P of high rigidity, fixed to the right-hand wall [16]. For $\mu < 0$ the system has the stable collisionless motion $q_\mu^*(t) = 1 - \mu - \cos t$ of period $T = 2\pi$, and when $\mu = 0$ this motion includes grazing collisions at $t = 0,\ \pm T,\ \pm 2T, \dots$.

The general solution of system (4.1) for $q > 0$ has the form

$$q(t) = q_\mu^*(t) + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{4.2}$$

where $\lambda_{1,2}$ are roots of the characteristic equation

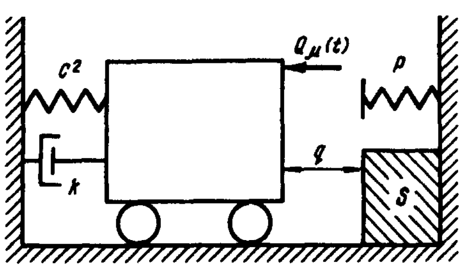

Fig. 1.

$$\lambda^2 + 2k\lambda + c^2 = 0$$

The matrix X_0 and its eigenvalues are

$$X_0 = \frac{1}{\lambda_2 - \lambda_1} \left\| \begin{matrix} \lambda_2\rho_1 - \lambda_1\rho_2 & \rho_2 - \rho_2 \\ o^2(\rho_1 - \rho_2) & \lambda_2\rho_2 - \lambda_1\rho_1 \end{matrix} \right\|$$

$$\rho_{1,2} = \exp(\lambda_{1,2}T), \quad x_{1,2} = (\rho_2 - \rho_1)/(\lambda_2 - \lambda_1)$$

We shall depict the periodic motions in the (μ, $q_{\min}$) plane, where $q_{\min}$ is the minimum value of the coordinate in a neighbourhood of the times $t = 0$, $\pm T$, $\pm 2T$, ... A motion of period T corresponds to one point on this plane, a motion of period $2T$ to a pair of points with the same abscissa, etc. Possible C-bifurcation scenarios are shown in Fig. 2, where the continuous lines correspond to stable motions, and the dashed lines to unstable ones: (a) is a "saddle-node", (b) is a subcritical "fork", (c) is a supercritical plus subcritical "fork", and (d) is a cascade of period-doublings. The theorem proved above enables us to identify the first of these cases, and the period-doubling type can be determined by the presence of periodic motions with collisions (1.3) when $\mu < 0$.

In case (a) T-periodic motions exist with one collision per period, i.e. the conditions

$$q(t_0) = q(t_0 + T) = 0, \quad \dot{q}(t_0) = -\kappa\dot{q}(t_0 + T) > 0 \tag{4.3}$$

are satisfied.

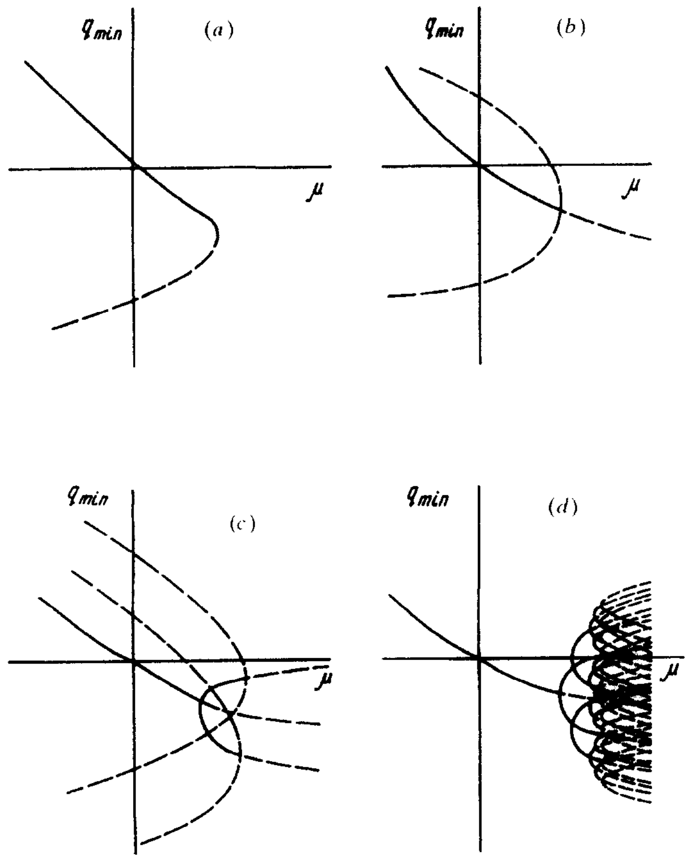

Fig. 2.

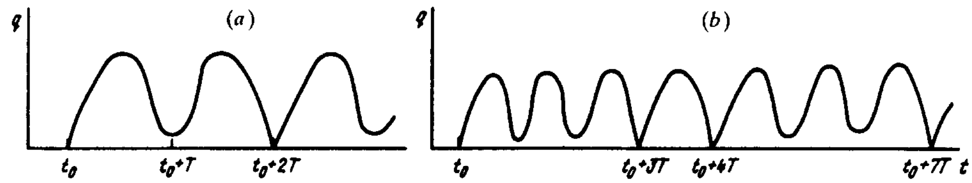

Fig. 3.

By (4.2) system (4.3) is linear in $C_{1,2}$. Eliminating these constants, we obtain

$$(1+\kappa)(\rho_2-\rho_1)\dot{q}_\mu^*(t_0)+q_\mu^*(t_0)[\lambda_1(1-\rho_2)(1+\kappa\rho_1)-\lambda_2(1-\rho_1)(1+\kappa\rho_2)]=0$$

$$\dot{q}(t_0)=q_\mu^*(t_0)(1-\rho_1)(1-\rho_2)\kappa(1+\kappa)^{-1}(\lambda_1-\lambda_2)(\rho_2-\rho_1)^{-1}$$

If $\mu<0$, then $q_\mu^*(t_0)>0$ and the sign of $\dot{q}(t_0)$ is opposite to the sign of the element x_{12} of matrix X_0. Consequently, case (a) arises under the condition $x_{12}<0$, which corresponds to the assertion of the theorem.

In case (b) when $\mu<0$ a $2T$-periodic motion exists with one collision per period, a typical graph being shown in Fig. 3(a). By analogy with Eqs (4.3) we form the system

$$q(t_0)=q(t_0+2T)=0,\quad q(t_0+T)>0,\quad \dot{q}(t_0)=-\kappa\dot{q}(t_0+2T)>0$$

which can be transformed to the form

$$q_\mu^*(t_0)(1-\rho_1^2)(1-\rho_2^2)\kappa(1+\kappa)^{-1}(\lambda_1-\lambda_2)(\rho_2^2-\rho_1^2)^{-1}>0 \tag{4.4}$$
$$q_\mu^*(t_0)(1-\rho_1)(1-\rho_2)(\rho_2+\rho_1)^{-1}<0$$

In case (c) there is a $4T$-periodic motion with two collisions per period, schematically shown in Fig. 3(b). The conditions for it to exist are as follows:

$$q(t_0)=q(t_1+3T)=0, q'(t_1+3T)=q'(t_0+4T)=0,\quad q(t_0+T)>0$$
$$q(t_0+2T)>0, \dot{q}(t_0)=-\kappa\dot{q}'(t_0+4T)>0, \dot{q}'(t_1+3T)=-\kappa\dot{q}(t_1+3T)>0$$
$$q'(t)=q_\mu^*(t)+C_1' e^{\lambda_1 t}+C_2' e^{\lambda_2 t}$$

from which we obtain

$$q_\mu^*(t_0)(\lambda_1-\lambda_2)(\rho_2^3-\rho_1^3)^{-1}>0,\quad q_\mu^*(t_0)[1-(1+\rho_1)(1+\rho_2)]<0 \tag{4.5}$$

One can similarly establish conditions under which C-bifurcations reduce to two super- and one subcritical "fork", etc.

Relations (4.3)–(4.5) have a simple interpretation in terms of eigenvalues. If the $\lambda_{1,2}$ are real, then the differences $(\lambda_1-\lambda_2)$ and $(\rho_1-\rho_2)$ have the same sign, because none of these groups of conditions is satisfied. In this case C-bifurcation leads to a doubling cascade (Fig. 2d), which is confirmed by numerical modelling [17].

If the numbers λ_1 and λ_2 are complex-conjugate, $\lambda_{1,2}=V\pm iW$, $V<0$, $W>0$, then $\rho_{1,2}=e^V(\cos W\pm i\sin W)$. Cases with the following values were investigated: (a) $\sin W<0$, (b) $\sin W>0$, $\sin 2W<0$, (c) $\sin W>0$, $\sin 2W>0$, $\sin 3W<0$.

One can draw the overall conclusion that for values of $k>c$ the oscillator (4.1) goes through a period-doubling cascade as a result of C-bifurcations. If however $k<c$, then stability is lost as

a result of "saddle-node" bifurcation or subcritical period-doubling (which can be preceded by several supercritical doublings).

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